CHAPTER 13

Grasping the Invisible Geometry Of Creation

In the previous chapter, we shifted our attention from the visible form of Ceres' orbit, to its *generation* in a higher domain. With the help of a simple geometrical metaphor, we represented the higher domain by a circular cone with its axis in the vertical direction, and the lower, "visible domain" by a horizontal plane. We made the plane intersect the cone at its vertex, at the location corresponding to the center of the sun, and likened the relationship of visible events to events in the higher, "conical space," to a projection from the cone, parallel to the conical axis, down to the plane.

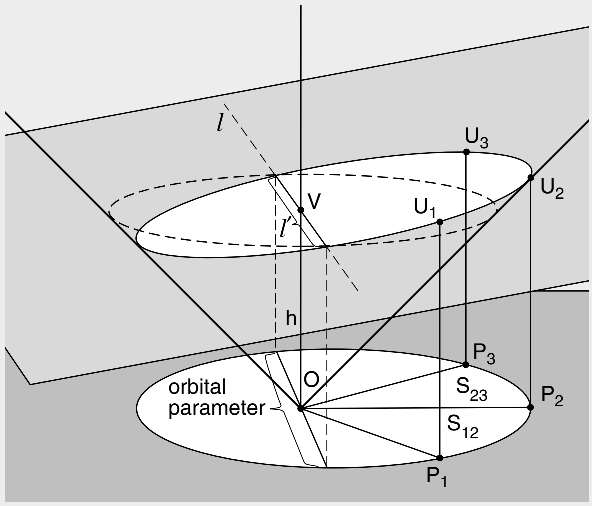
In fact, if we trace Ceres' orbit on the horizontal plane, that form is the projected image of a conic section on the cone.

How is it possible to use the geometry of visual space, to "map" relationships in a "higher space" of an axiomatically different character? Only as paradox. Obviously, no "literal" representation is possible, nor do we have a mere analogy in mind. When we represent visual space by a *two-dimensional* plane (inside visual space!), and the higher space as a cone in "three dimensions," projected onto the plane, we do not mean to suggest that the higher space is only "higher" by virtue of its having "more dimensions." Rather, we should "read" the axis of the cone in our representation, to signify a different *type* of ordering principle than that of visual space—one embodying features of the transfinite, "anti-entropic" ordering of the Universe as a whole.

Reflecting on the irony of applying constructions of elementary geometry to such a metaphorical mapping, the following idea suggests itself: The geometry of visible space has shown itself *appropriate* to a process of discovery of the reality lying outside visual space, when it is considered not as something fixed and static, but as constantly *redefined* and *developed* by our cognitive activity, just as we develop the well-tempered system of music through Classical thorough-composition. Should we not treat elementary geometry from the standpoint, that visual space is created and "shaped" *to the purpose* of providing reason with a pathway toward grasping the "invisible geometry" of Creation itself?

Keeping these ironies in mind, let us return to the challenge which last chapter's discussion placed in front of us. We developed a method for constructing an approximation of Ceres' position, which did not adequately take into account the space-time curvature in the small. As a result, we introduced a source of error which could lead to major discrepancies between our estimate of the Earth-Ceres distance, and the real distance. If Gauss had not corrected that fault, his attempt at forecasting the orbit of Ceres, would have been a failure.

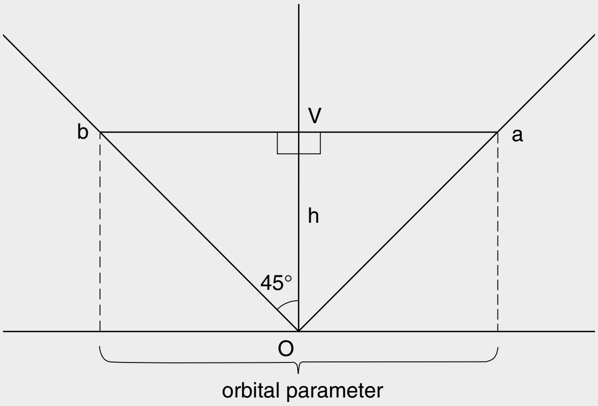
FIGURE 13.1. *The orbital parameter is the projection of diameter l' in the circular cross-section of the cone at height h of V. Diameter l' is generated by the intersection of the circular and elliptical cross-sections.*



We have no alternative, but to investigate the curvature in the small which characterizes the spatial relationship between any three positions P_1, P_2, P_3 of a planet, solely by virtue of the fact that they are "moments" of one and the same Keplerian orbit. And, to do that without any assumption concerning the particular form of the conic-section orbit.

We projected the three given positions up to the cone, to obtain points U_1, U_2, U_3. The latter three points determine a *unique* plane, which intersects the cone in a conic section, and whose projection onto the horizontal plane is the visible form of Ceres' orbit. The intersection of that same plane with the axis of the cone, at a point we called *V,* is an important singularity. The circular cross-section of the cone at the "height" of *V*, is cut by the $U_1U_2U_3$

FIGURE 13.2. *Relationship of height h to the orbital parameter. The diagram represents the cross-sectional "cut" of the cone, by the plane defined by the vertical axis and the segment l′ (represented here as the segment between points a and b). Since the apex angle of the cone is 90°, the triangles aVO and bVO are isosceles right triangles. Consequently, h = aV = bV = (1/2) × (length of ab) = half-parameter of orbit.*



plane at two points, which are the endpoints of a diameter l' through V. That diameter projects (without change of length) to the segment which represents the width of the Ceres' orbit, measured perpendicularly to the axis of the orbit at its focus O. That length is what Gauss calls the "orbital parameter." **(Figure 13.1)**

Thus, Gauss's parameter is equal to the cross-section diameter of the cone at V, which, in turn, is proportional to the height of V on the conical axis. The factor of proportionality depends upon the apex angle of the cone; that factor becomes equal to 1, if we choose the apex angle of the cone to be 90° (so that the surface of the cone makes an angle of 45° with the horizontal plane at O). Let us choose the apex angle so. In that case, *the height h of V above the axis is equal to half the orbital parameter.* **(Figure 13.2)**

Now recall, that according to Gauss's recasting of Kepler's constraints, the area swept out by the planet in any time interval, is proportional to the elapsed time, multiplied by the *square root of the half-parameter.* (SEE Chapter 8) Our analysis actually showed, that the constant of proportionality is π, when the elapsed time is measured in years, length in Astronomical Units (A.U.) (Earth-sun distance), and area in square A.U.

From these considerations, we can now express the areas of the orbital sectors of Ceres, in terms of the elapsed times and the height h of V on the cone. For example:

$$S_{12} = \sqrt{h} \times \pi \times (t_2 - t_1)\,, \quad \text{and}$$
$$S_{23} = \sqrt{h} \times \pi \times (t_3 - t_2)\,.$$

At the close of the last chapter, we remarked that the value of h must somehow be expressible in terms of the triangular areas T_{12}, T_{23}, T_{13}; and, that the resulting link with S_{12} and S_{23}, via h, would finally provide us with a much more "fine-tuned" approximation to the crucial ratios $T_{12}:T_{13}$ and $T_{23}:T_{13}$ than was possible on the basis of our initial, crude approach. **(Figure 13.3)**

Not to lose your conceptual bearings at this point, before we launch into a crucial battle, remember the following: The significance of the orbital parameter, now represented by h, lies in the fact that it embodies the relationship between

(i) the Keplerian orbit *as a whole*;

(ii) the array of "geometrical intervals" between any three positions P_1, P_2, P_3 on the orbit; and

(iii) the curvature of each arbitrarily small "moment of action" in the planet's motion, as expressed in the corresponding orbital sector, and above all in the relationship between the "curved" sectoral area and corresponding triangular area.

Gauss focussed his attention on the sector and triangle formed between the first and the third positions, S_{13} and T_{13}. Our experimental calculations, reviewed in the last chapter, indicated that the discrepancy between these two, is the main source of error in our method for calculating the Earth-Ceres distance. Hence, Gauss looked for a way to accurately estimate that area.

Gauss noted that *most* of the excess of S_{13} over T_{13}, i.e., the lune-shaped area between the orbital arc from P_1 to P_3 and the segment P_1P_3, is constituted by the triangular

FIGURE 13.3. *Our conical projection, which contains both the triangular areas and the elliptical sectors as well as the orbital parameter h, will help us to devise a "fine-tuned" approximation to the crucial coefficients required to determine the orbit of Ceres (cf. Figure 13.1).*

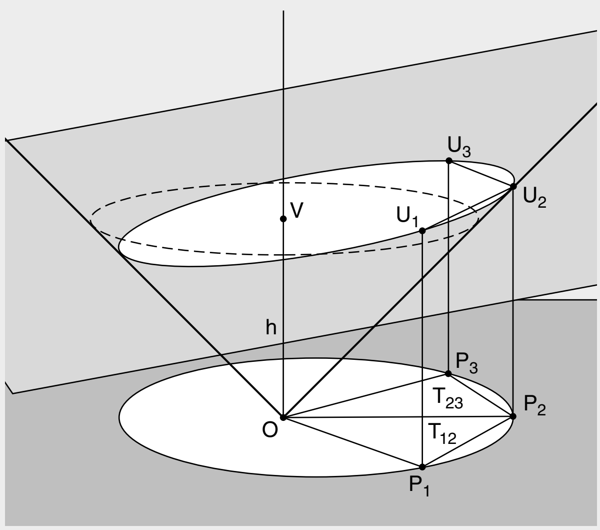
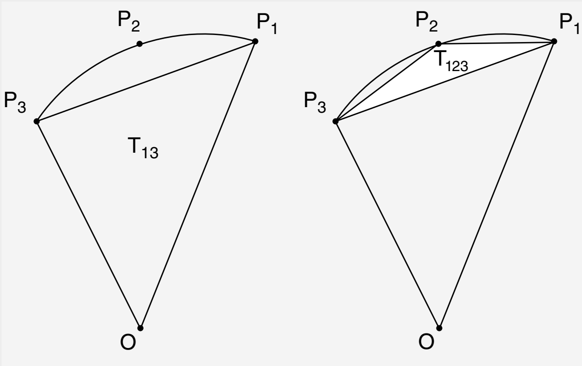


FIGURE 13.4. *Most of the excess of S_{13} over T_{13}, which is the lune-shaped area, is constituted by triangle T_{123}.*



area $P_1P_2P_3$. Denote this triangle—the triangle formed between all three positions of the planet—by "T_{123}." Gauss also observed, that T_{123} is the excess of T_{12} and T_{23} combined, minus T_{13}. **(Figure 13.4)**

How will our exploration of conical geometry help us to get a grip on that little "differential" T_{123}? We voiced the expectation, earlier, that "the height *h* of *V* on the cone must somehow be expressible in terms of the triangular areas T_{12}, T_{23}, T_{13}." The time has come, to make good on our promise.

An Elementary Proposition of Descriptive Geometry

Those brought up in the geometrical culture of Fermat, Desargues, Monge, Carnot, and Poncelet would experience no difficulty whatever at this point. But, most of us today, emerged from our education as geometrical illiterates.* With a bit of courage, however, this condition can be remedied.

Recall how we used the triangular areas T_{12}, T_{13}, and T_{23} to measure the relationship between the Ceres position P_2 and P_1, P_3, as a combination of displacements along the axes OP_1 and OP_3. Evidently, we touched upon a principle of geometry relevant to a much broader domain.

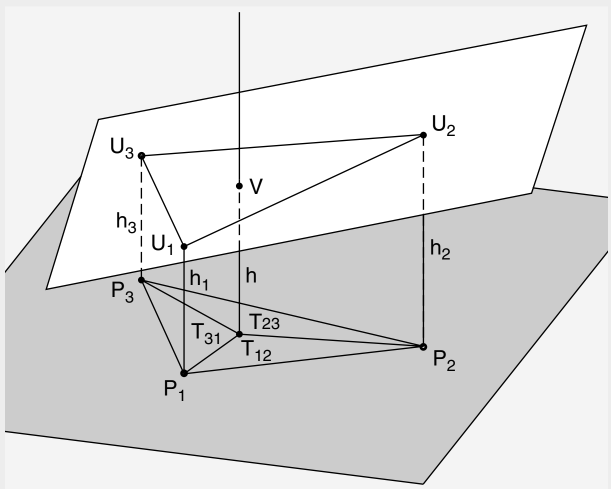
The nature of the relationship we are looking for now, becomes most clearly apparent, if we put Piazzi's observations aside for the moment, and examine, instead, the hypothetical case, where the P_1, P_2, P_3 are widely separated—say, at roughly equal angles (i.e., roughly 120° apart) around *O*. **(Figure 13.5)** In this case, we have a triangle $P_1P_2P_3$ in the horizontal plane, which contains the point *O* and is divided up by the radial lines OP_1, OP_2, OP_3 into the smaller triangles T_{12}, T_{23}, and T_{31}. Above the triangle $P_1P_2P_3$, and projecting exactly onto it, we have the triangle $U_1U_2U_3$. This latter triangle "sits on stilts," as it were, over the former. The "stilts" are the vertical line segments P_1U_1, P_2U_2, and P_3U_3, whose heights are h_1, h_2, and h_3. Point *V* is the place where the axis of the cone passes through triangle $U_1U_2U_3$. How does the height of *V* above the horizontal plane, depend on the heights h_1, h_2, and h_3?

This is an easy problem for anyone cultured in synthetic geometry, rather than the stutifying, Cartesian form of textbook "analytical geometry" commonly taught in schools and universities. The approach called for here, is exactly the opposite of "Cartesian coordinates." Don't treat the array of positions, and the organization of space in general, as a dead, static entity. Think, instead, in *physical terms;* think in terms of change, displacement, work. For example: What would happen to the height of *V*, if we were to *change* the height of one of the points U_1, U_2, U_3?

Suppose, for example, we keep U_2 and U_3 fixed, while raising the height of U_1 by an arbitrary amount "*d*," raising it in the vertical direction to a new position U_1'. **(Figure 13.6)** The new triangle $U_1'U_2U_3$ intersects the axis of the cone at a point V', higher than *V*. Our immediate task is to characterize the functional relationship between the parallel vertical segments VV' and U_1U_1'.

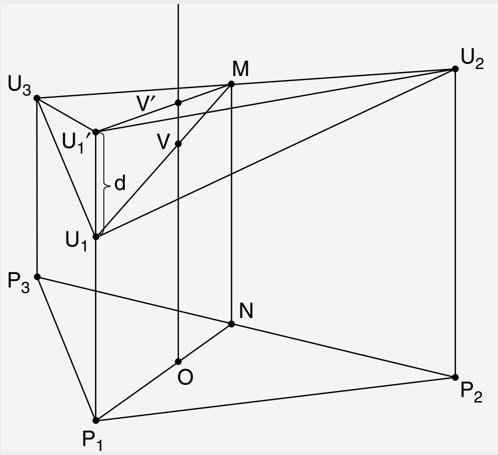
The two triangles $U_1U_2U_3$ and $U_1'U_2U_3$ share the common side U_2U_3, forming a wedge-like figure. Cut that figure by a vertical plane passing through the segments VV' and U_1U_1'. The intersection includes the segment U_1U_1', and the lines through U_1 and *V*, and

FIGURE 13.5. *How does the height h of V depend upon heights h_1,h_2,h_3, which are in turn a function of the position of the plane through U_1,U_2,U_3?*



* Including the present author, incidentally.

FIGURE 13.6. *Tilt the plane of the U's up from U_1 to U_1', to generate V'. What is the functional relationship between segments UU' and VV'?*



through U_1' and V', respectively, which meet each other at some point M on the segment U_2U_3. Two triangles are formed in the vertical plane from those vertices: U_1MU_1', and a sub-triangle VMV'. Given that VV' is parallel to U_1U_1', those two triangles will be similar to each other.

The ratio of similarity of these triangles, determines the relationship of immediate interest to us, namely, that between the change in height of V (i.e., the length of VV') and the change in the height of U_1 (i.e., the length of U_1U_1').

To determine the ratio of similarity of the triangles, we need only establish the proportionality between any pair of corresponding sides. So, look at the ratio $MV:MU_1$, i.e., the ratio by which V divides the segment MU_1. *That ratio is not changed* when we project the segment onto the plane of P_1, P_2, P_3. Under the projection, U_1 projects to P_1, V projects to O, and M projects to some point N on the line P_2P_3.

Our problem is reduced to determining the ratio by which O divides the line segment NP_1—that latter being the projected image of the segment MU_1. Very simple! Look at P_2P_3 as the base of the triangle $P_1P_2P_3$. **(Figure 13.7)** Draw the line parallel to P_2P_3 through P_1. The distance separating that line and P_2P_3 is called the *altitude* of the triangle $P_1P_2P_3$, whose product with the length of the base, P_2P_3, is equal to twice the area of the triangle T_{123}. Next, draw the parallel to P_2P_3 through the point O. The gap between that line and P_2P_3, is the altitude of the triangle OP_2P_3, whose product with the length of the base P_2P_3 is equal to twice the area of triangle T_{23}.

Thus, the ratio of the distances between the first and second, and the first and third—that is, of the distances between P_2P_3 and each of the two lines parallel to it—is equivalent to the ratio of T_{23} to T_{123}. But, the ratio of distances between those parallels is "reproduced" in the proportion of the segments, formed on any line which cuts across all three. Taking in particular the line through O and P_1 (which intersects P_2P_3 at N) we conclude that

$$NO:NP_1::T_{23}:T_{123}.$$

By "inverse projection," the same holds true for the ratio of MV and MU_1, and by similarity, also for the ratio between VV' and U_1U_1'.

Our job is essentially finished. We have found, that when the height of U_1 is changed by any amount "d," the height of V changes by an amount whose ratio to d is that of T_{23} to T_{123}. In other words, the change in height of V will be $d \times (T_{23}/T_{123})$; or, to put still another way,

$$T_{123} \times \text{change of height of } V$$
$$= T_{23} \times \text{change of height of } U_1 .$$

What happens, then, if we start off with all the heights equal to zero, and raise the heights of the vertices, one at a time, to the given heights h_1, h_2, h_3, respectively? Raising U_1 from height zero to h_1, will increase the height of V, from zero to $h_1 \times (T_{23}/T_{123})$. Next (by the same reasoning, applied to U_2 instead of U_1), raising U_2 to the height h_2, will increase the height of V by an additional amount equal to $h_2 \times (T_{31}/T_{123})$.

Finally, raising U_3 to the height h_3, will raise V by an additional amount $h_3 \times (T_{12}/T_{123})$. The final height h of V, will therefore be equal to

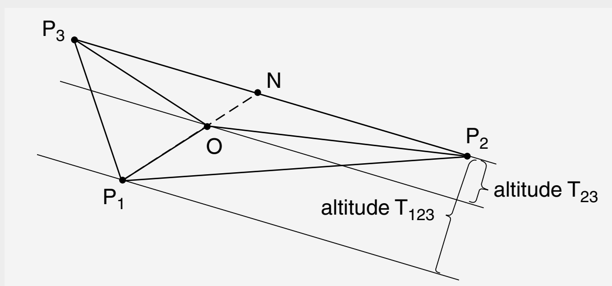
$$[h_1 \times (T_{23}/T_{123})] + [h_2 \times (T_{31}/T_{123})] + [h_3 \times (T_{12}/T_{123})],$$

or, in other words, $h \times T_{123}$ is equal to

$$(h_1 \times T_{23}) + (h_2 \times T_{31}) + (h_3 \times T_{12}) .$$

All of this referred to the case where points P_1, P_2, P_3 are separated by such large angles, that O lies within triangle $P_1P_2P_3$ ($=T_{123}$). In the actual case before us, the

FIGURE 13.7. *The division of segment NP_1 by point O is proportional to the ratio of the areas of triangles T_{23} and T_{123}.*



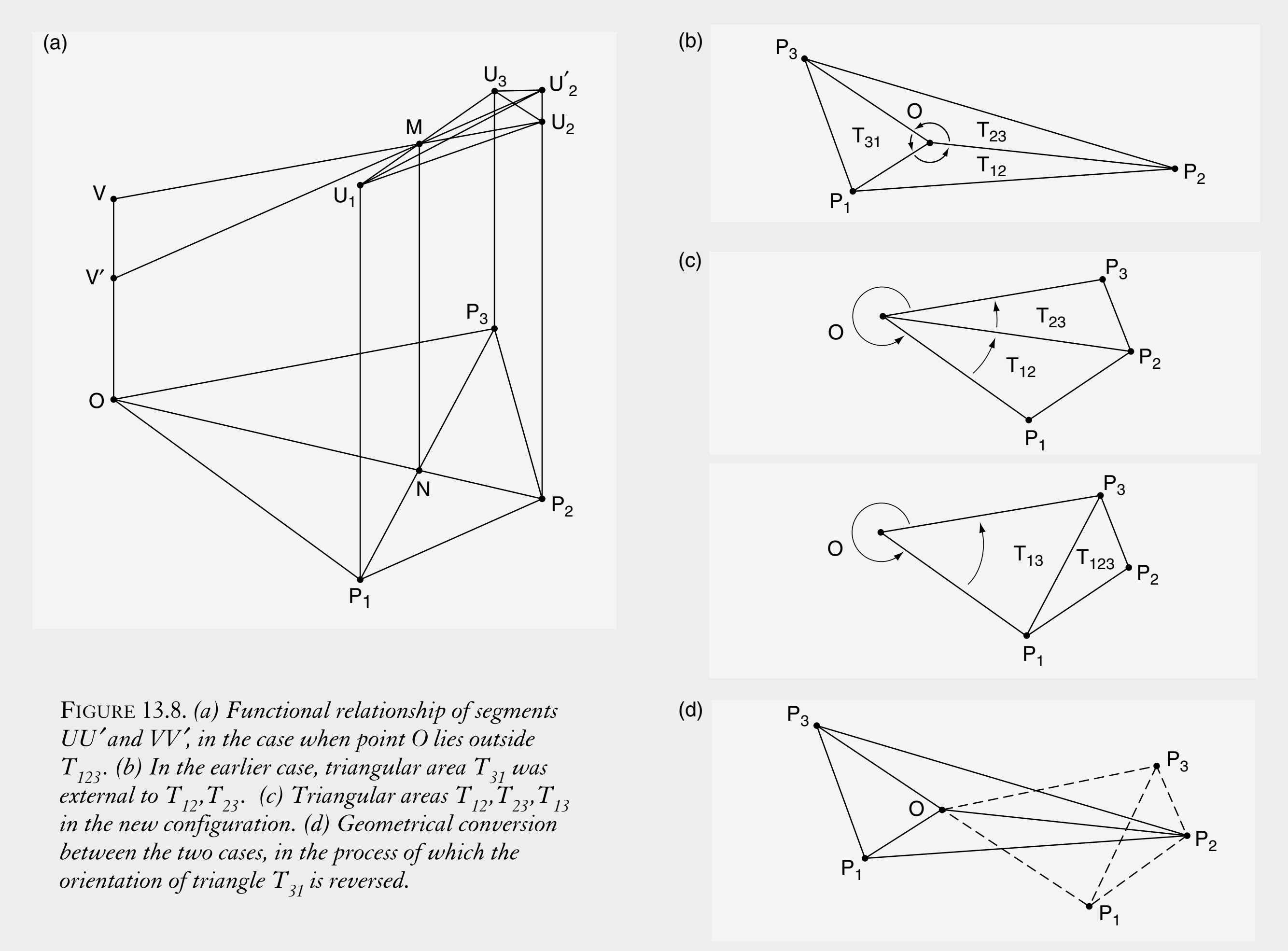


FIGURE 13.8. *(a) Functional relationship of segments UU′ and VV′, in the case when point O lies outside* T_{123}. *(b) In the earlier case, triangular area* T_{31} *was external to* T_{12}, T_{23}. *(c) Triangular areas* T_{12}, T_{23}, T_{13} *in the new configuration. (d) Geometrical conversion between the two cases, in the process of which the orientation of triangle* T_{31} *is reversed.*

triangle T_{123} is very small, and O lies outside it. **(Figure 13.8a)** Nevertheless, it is not difficult to see—and the reader should carry this out as an exercise—that nothing essential is changed in the fabric of relationships, except for one point of elementary *analysis situs*: We were careful to observe a consistent ordering in the vertices and the triangles, corresponding to rotation around O in the direction of motion of the planet. In keeping with this, "T_{31}" referred to the triangle whose angle at O is the angle swept out in a *continuing rotation,* from P_3 back to P_1. **(Figure 13.8b)** In our present case, where O lies outside triangle $P_1P_2P_3$ and the displacements from P_1 to P_2 and P_2 to P_3 are very small, the angle of that rotation is nearly 360°. **(Figure 13.8c)** In mere form, the resulting triangle OP_3P_1 is the same as OP_1P_3, and the areas T_{31} and T_{13} both refer to the same form; however, their *orientations* are different. **(Figure 13.8d)**

As Gauss emphasized in his discussions of the *analysis situs* of elementary geometry, our accounting for areas must take into account the differences in orientation, so the proper value to be ascribed to T_{31} must be the same magnitude as T_{13}, but with the *opposite sign*. In other words, $T_{31} = -T_{13}$. Examining the constructions defining the functional dependence of h on h_1, h_2, and h_3, for the case where the angle from P_3 to P_1 is more than 180°, we find that this *change of sign* is indeed necessary, to give the correct value for the contribution of the height of U to the height of V, namely, $h_2 \times -(T_{13}/T_{123})$. In fact, when we raise U_2, the height of V is *reduced*. For that reason the relationship of the areas and heights, in the case of the three positions of Ceres, takes the form

$$h \times T_{123} = (h_1 \times T_{23}) - (h_2 \times T_{13}) + (h_3 \times T_{12}),$$

or,

$$T_{123} = \frac{(h_1 \times T_{23}) - (h_2 \times T_{13}) + (h_3 \times T_{12})}{h}.$$

This is a starting point for evaluating the "triangular differential" T_{123}, which measures the effect of the space-time curvature in the small.

—JT

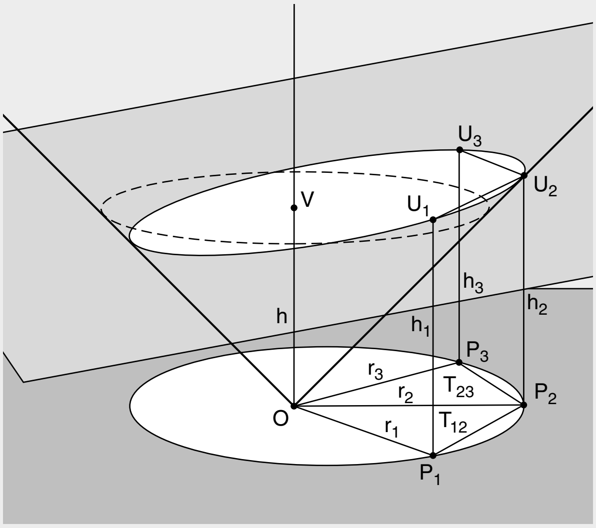
CHAPTER 14

On to the Summit

If our several-chapters' journey of rediscovery has often seemed like climbing a steep mountain, then this chapter will take us to the summit. From there, the rest of Gauss's solution will lie below us in a valley, easily surveyed from the work we have already done.

The crux of Gauss's approach, throughout, lies in his focussing on the relationship between what we have called the "triangular differential" formed between any three positions of a planet in a Keplerian orbit, and the physical characteristics of the orbit as a whole. (**Figure 14.1**)

FIGURE 14.1. *Gauss has focussed on the relationship between the orbital sectors, the triangular areas, the orbital parameter (which is equal to the height of V), and the characteristics of the orbit as a whole.*



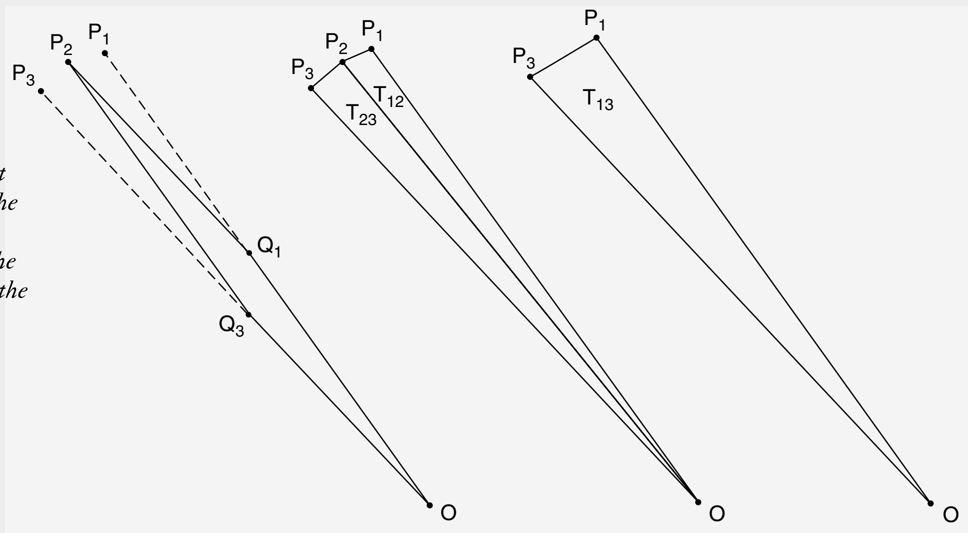
That relationship is implicit in the Gauss-Kepler constraints, and particularly in the "area law," according to which the areas swept out by the planet's motion between any two positions, are proportional to the corresponding elapsed times.

Recall our first pathway of attack on the Ceres problem. It was based on the observation, that the area of the orbital sector between any two of the three given positions, is only slightly larger than the triangular area, formed between the same two positions (and the center of the sun). On the other hand, we found that the values of those same triangular areas—or, rather, the ratios between them—determined the spatial relationship between the three Ceres positions, as expressed in terms of the "parallelogram law" of displacements. (**Figure 14.2**) We discovered a method for determining the positions of Ceres (or at least one of them), given the values of the triangular ratios, by applying those values to the known positions of the Earth, adducing a discrepancy resulting from the difference in curvature

FIGURE 14.2. *In Chapter 10, we found that the intermediate position P_2 of Ceres can be related to the other two positions P_1,P_3 in the following way: P_2 is the resultant of a combination (according to the "parallelogram law") of two displacements OQ_1, OQ_2 along the axes OP_1 and OP_2, respectively, the positions of Q_1 and Q_3 being determined by the relationships*

$$\frac{OQ_1}{OP_1} = \frac{T_{23}}{T_{13}}, \text{ and}$$

$$\frac{OQ_3}{OP_3} = \frac{T_{12}}{T_{13}}.$$



between the Earth and Ceres orbits, and then reconstructing Ceres' position from that discrepancy by a kind of "inverse projection." (**Figure 14.3**)

The obvious difficulty with our method, lay in the circumstance, that we had no *a priori* knowledge of the exact ratios of triangular areas, required to carry out the construction. At that point, we could only say that the ratios must be "fairly close" to the ratios of the corresponding orbital sectors, whose values we know to be equal to the ratios of the elapsed times according to the "area law." Our first inclination was to try to ignore the difference between the triangular and sectoral areas, and to apply the known ratios of elapsed times to obtain an approximate position for the planet. Unfortunately, a closer analysis of the effect of any given error on the outcome of the construction, showed that the slight discrepancy between triangles and sectors can produce an unacceptable final error of 20 percent, or even more (depending on the actual dimensions of Ceres' orbit). This left us with no alternative, but to look for a new principle, allowing us to estimate the magnitude of the difference between the curvilinear sectors and their triangular counterparts.

We noted, as Gauss did, that the largest discrepancy occurs in the case between the first and third positions, P_1 and P_3, which are the farthest apart. Comparing sector S_{13} with triangle T_{13}, the difference between the two is the lune-shaped area between the orbital arc and the chord connecting P_1 and P_3. (**Figure 14.4**) *Most* of that area belongs to the triangle formed between P_1, P_3 and the intermediate position P_2, a triangle we designated T_{123}. Gauss realized, that the key to the whole Ceres problem, is to get a grip on the magnitude of that "triangular differential," which expresses the effect of the curvature of Ceres' orbit over the interval spanned by the three positions. This "local" curvature reflects, in turn, the characteristics of the entire orbit.

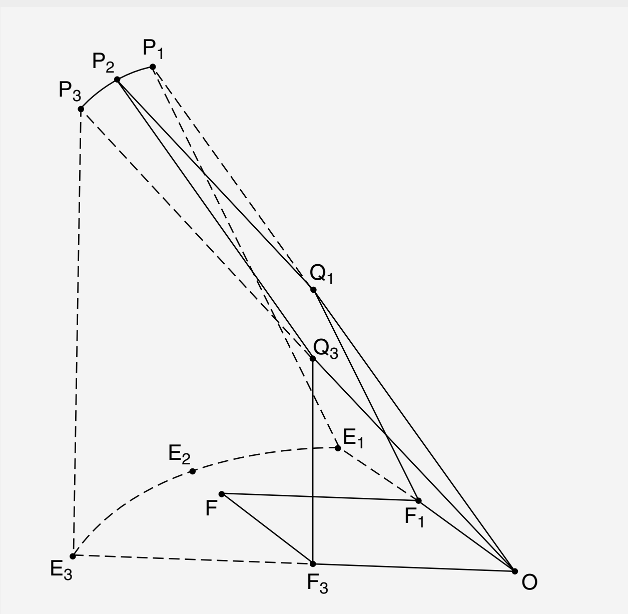
Given the multiple, interconnected variabilities embodied in the notion of an arbitrary conic-section orbit, we cannot expect a simple, linear pathway to the required estimate. We must be prepared to carry out a somewhat extended examination of the array of geometrical factors which combine to determine the magnitude of T_{123}. Our strategy will be to try to map the essential feature of that interconnectedness, in terms of a relationship of *angles* on a single circle.

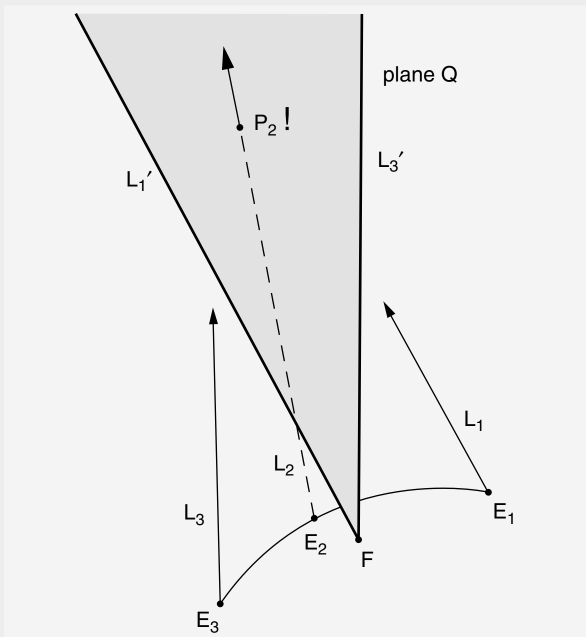
In doing so, we are free to make use of simple special cases and numerical examples, as "navigational aids" to guide our search for a general solution.

Accordingly, look first at the simplified, hypothetical case of a circular orbit. In that case, the planet's motion is uniform; the angles swept out by the radial lines to the sun are proportional to the corresponding elapsed times, divided by the total period T of the orbit. According to Kepler's laws, $T^2 = r^3$, so T is equal to the three-halves power of the circle's radius ($r^{3/2}$).

At first glance the area T_{123} is a somewhat complicated

FIGURE 14.3. *In Chapter 11, we located Ceres' position P_2 on plane Q, using a construction pivoted on the discrepancy between the curvatures of the orbits of Earth ($E_1E_2E_3$) and Ceres ($P_1P_2P_3$) .*





function of the angles at the sun. But there is an underlying harmonic relationship expressed in a beautiful theorem of Classical Greek geometry, which says that *the area of a triangle inscribed in a circle, is equal to the product of the sides of the triangle, divided by four times the circle's radius.* (**Figure 14.5**) Applying this to our case, the area T_{123} is equal to the product of the *chords* P_1P_2, P_2P_3, and P_1P_3, divided by four times the orbital radius. **(Figure 14.6)**

Now, to a first approximation, when the planet's positions P_1, P_2, P_3 are not too far apart, the length of each such chord is very nearly equal to the corresponding arc on the circle. The latter, in turn, is equal in length to the total circumference of the circle, times the ratio of the elapsed time for the arc to the full period of the circular orbit [i.e., $2\pi r \times$ (elapsed time$/r^{3/2}$)]. Applying this, we can estimate T_{123} by routine calculation as follows:

$$T_{123} \simeq \frac{1}{4r}(P_1P_2 \times P_2P_3 \times P_1P_3)$$

$$= \frac{1}{4r} \times \left[2\pi r \times \left(\frac{t_2 - t_1}{r^{3/2}}\right)\right]$$

$$\times \left[2\pi r \times \left(\frac{t_3 - t_2}{r^{3/2}}\right)\right] \times \left[2\pi r \times \left(\frac{t_3 - t_1}{r^{3/2}}\right)\right]$$

$$= 2\pi^3 \times \frac{(t_2 - t_1) \times (t_3 - t_2) \times (t_3 - t_1)}{r^{5/2}}$$

(the $\simeq$ symbol means "approximately equal to").

What is of interest here, is not the details of the calculation, but only the general form of the result, which is to approximate T_{123} by a simple function of the elapsed times and one additional parameter (the radius). Can we develop a similar estimate for T_{123}, without making any assumption about the specific shape of the Keplerian orbit? It is a matter of evoking the higher, relatively constant curvature, which governs the variable curvatures of non-circular orbits. Gauss had reason to be confident, that, on the basis of his method of hypergeometrical and modular functions, and guided by numerical experiments on known orbits, he could develop the required estimate—one in which the role of the radius in a circular orbit, would be played by some combination of the sun-Ceres distances for P_1, P_2, P_3.

FIGURE 14.4. *The lune-shaped difference between S_{13} and T_{13} is largely constituted by triangle T_{123}.*

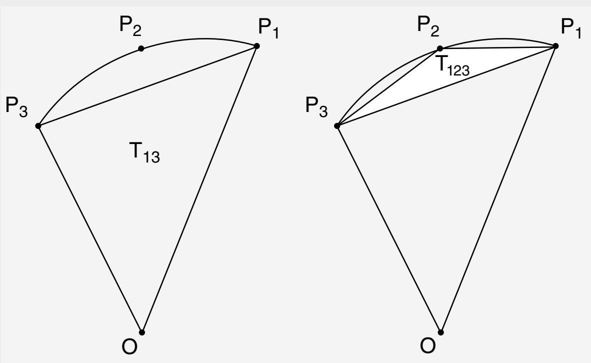


FIGURE 14.5. *Classical theorem of Greek geometry: The area of any triangle ABC inscribed in a circle, is equal to (AB × BC × CA)/4r, where AB,BC,CA are the chords forming the sides of the triangle, and r is the radius.*

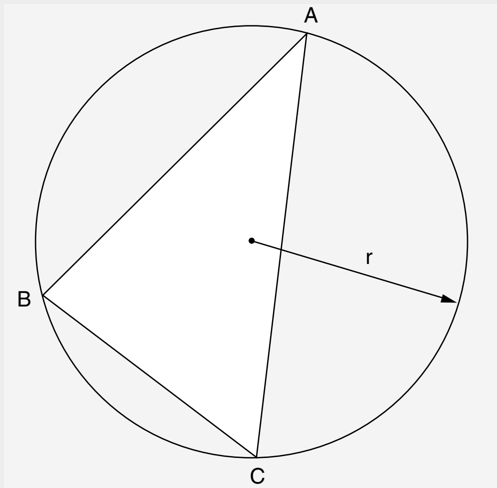
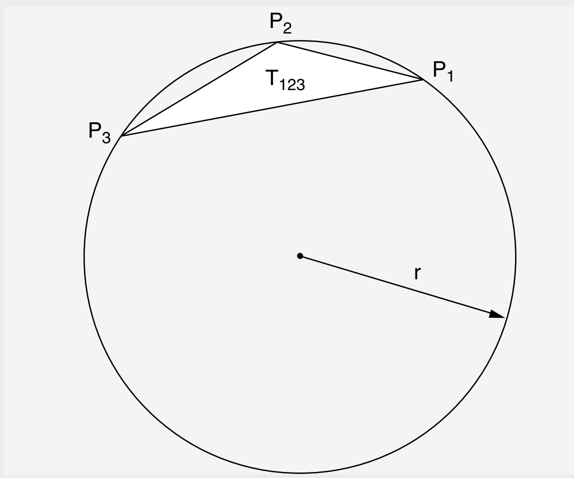


FIGURE 14.6. *Apply the Classical theorem to triangle T_{123}: area $T_{123} = (P_1P_2 \times P_2P_3 \times P_1P_3)/4r$.*



Nevertheless, a worrying thought occurs to us at this point: What use is a whole elaborate investigation concerning T_{123}, if the result ends up depending on an unknown, whose determination is the problem we set out to solve in the first place? The sun-Ceres distance, is no less an unknown than the Earth-Ceres distance; in fact, each can be determined from the other, by "solving" the triangle between the Earth, Ceres, and the sun, whose angle at the "Earth" vertex is known from Piazzi's measurements. **(Figure 14.7)** But, if neither of them are known, what use is the triangular relationship? And if, as it looks now, the necessary correction to our initial, crude approach to calculating the Earth-Ceres distance, turns out to depend upon a foreknowledge of that distance, then our whole strategy seems built on sand.

But, don't throw in the towel! Perhaps, by *combining* the various relationships and estimates, and using one to correct the other in turn, we can devise a way to rapidly "close in" on the precise values, by a "self-correcting" process of successive approximations. This, indeed, is exactly what Gauss did, in a most ingenious manner.

Before getting to that, let's dispense with the immediate task at hand: to develop an estimate for the "differential" T_{123}, independently of any *a priori* hypothesis concerning the shape of the orbit.

As already mentioned, the task in front of us involves a multitude of interconnected variabilities, which we must keep track of in some way. Although these variabilities are in reality nothing but facets of a single, organic unity, a certain amount of mathematical "bookkeeping" appears unavoidable in the following analysis, on account of the relative linearity of the medium of communication we are forced to use. Contrary to widespread prejudices, there is nothing sophisticated at all in the bookkeeping, nor does it have any content whatsoever, apart from keeping track of an array of geometrical relationships of the most elementary sort. The sophisticated aspect is implicit, "between the lines," in the Gauss-Kepler hypergeometric ordering which shapes the entire pathway of solution.

The essential elements are already on the table, thanks to last chapter's work on the conical geometry underlying the orbit of Ceres. Our investigation of the relationship between the triangular areas T_{12}, T_{23}, T_{13}, and T_{123}, the heights of points on the cone corresponding to P_1, P_2, P_3, and Gauss's orbital parameter h, yielded a conclusion which we summarized in the formula

$$T_{123} = \frac{(h_1 \times T_{23}) - (h_2 \times T_{13}) + (h_3 \times T_{12})}{h} \qquad \textbf{(1)}$$

(shown in Figure 14.1).

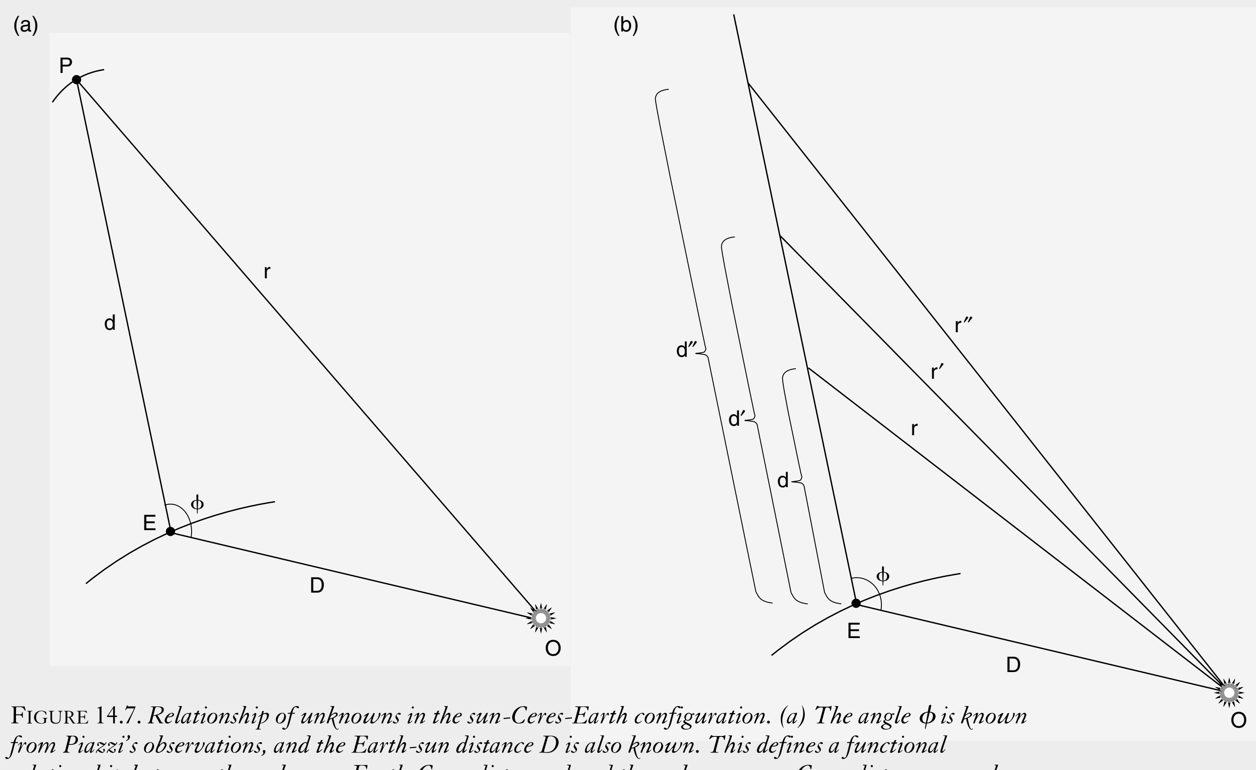


FIGURE 14.7. *Relationship of unknowns in the sun-Ceres-Earth configuration. (a) The angle ϕ is known from Piazzi's observations, and the Earth-sun distance D is also known. This defines a functional relationship between the unknown Earth-Ceres distance d and the unknown sun-Ceres distance r, as shown in (b). (b) To each hypothetical value of r, there corresponds a unique value of d, consistent wth the known values of ϕ and D.*

Two immediate observations on this account: First, recall our choice of 90° for the apex angle of the cone. Under that condition, the heights h_1, h_2, h_3 will be the same as the radial distances of P_1, P_2, P_3 from the sun. We shall denote the latter r_1, r_2, r_3.

Secondly: According to the Kepler-Gauss constraints, the *square root* of the half-parameter is proportional to the ratio of the sectoral areas swept out to the elapsed times. (SEE Chapter 8) We also determined the constant of proportionality, which amounts to multiplying the elapsed time by a factor of π. The *half-parameter itself* will then be equal to the quotient of the *product* of the areas swept out in any given *pair* of time intervals, divided by π^2 times the product of the corresponding elapsed times. So, for example, we can combine the relationships

$$\sqrt{h} = \frac{S_{12}}{(t_2-t_1) \times \pi} \quad ,$$

$$\sqrt{h} = \frac{S_{23}}{(t_3-t_2) \times \pi}$$

(by multiplying), to get

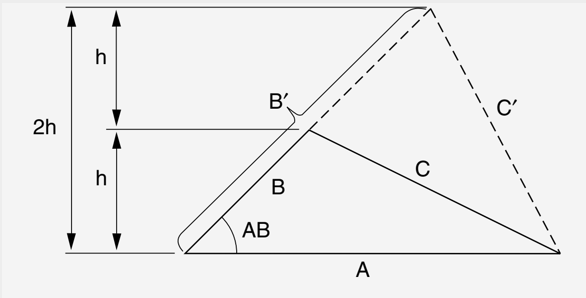
$$h = \frac{S_{12} \times S_{23}}{(t_2-t_1) \times (t_3-t_2) \times \pi^2} . \qquad \textbf{(2)}$$

This, according to **Equation (1)** above, is the magnitude by which we must divide $(r_1 \times T_{23}) - (r_2 \times T_{13}) + (r_3 \times T_{12})$, to obtain the value of the "triangular differential" T_{123}.

With that established, take a careful look at the combination of the radii r_1, r_2, r_3 and the triangular areas T_{23}, T_{13}, and T_{12}, entering into the value of T_{123}. Those triangular areas are determined by the array of vertex angles at the sun, i.e., the angles formed by the radial sides OP_1, OP_2, OP_3, together with the values of r_1, r_2, r_3 which measure the lengths of the sides. These are all interconnected, by virtue of the fact that P_1, P_2, P_3 lie on one and the same conic section. Let us try to "crystallize out" the kernel of the relationship, by focussing on the angles and attempting to "project" the entire array in terms of relationships within a single *circle***.**

There is a simple relationship between area and sides of a triangle, which can help us here. If we multiply one side of a triangle by any factor, while keeping an adjacent side and the angle between them unchanged, then the area of the triangle will be multiplied by the same factor. So, for example, if we double the length of the side B in a triangle with sides *A, B, C,* while keeping the length of A and the angle AB constant, then the resulting triangle of sides *A, 2B,* and some length C', will have an area equal to twice that of the original triangle. (**Figure 14.8**) The reason is clear: Taking A as the base, doubling B increases the altitude of the original triangle by the same factor, while the base remains the same. Hence the area—which is equivalent to half the base times the altitude—will also be doubled. Similarly for multiplying or dividing by any other proportion.

FIGURE 14.8. *Doubling a side of a triangle, while keeping the adjacent side and angle constant, doubles the area of the triangle.*



Applying this to T_{23}, for example, notice that its longer sides are radial segments from the sun, having lengths r_2 and r_3. (**Figure 14.9a**) If we divide the first side by r_2 and the second side by r_3, then we get a triangular area T_{23}', whose corresponding sides are now of unit length, and whose area is T_{23} divided by the product of r_2 and r_3. Turning that around, the area T_{23} is equal to $r_2 \times r_3 \times T_{23}'$. The product $r_1 \times T_{23}$, which enters into our calculation of the "triangular differential," is therefore equal to $r_1 \times r_2 \times r_3 \times T_{23}'$.

The same approach, applied to T_{13}, yields the result that

$$T_{13} = r_1 \times r_3 \times T_{13}' ,$$

and

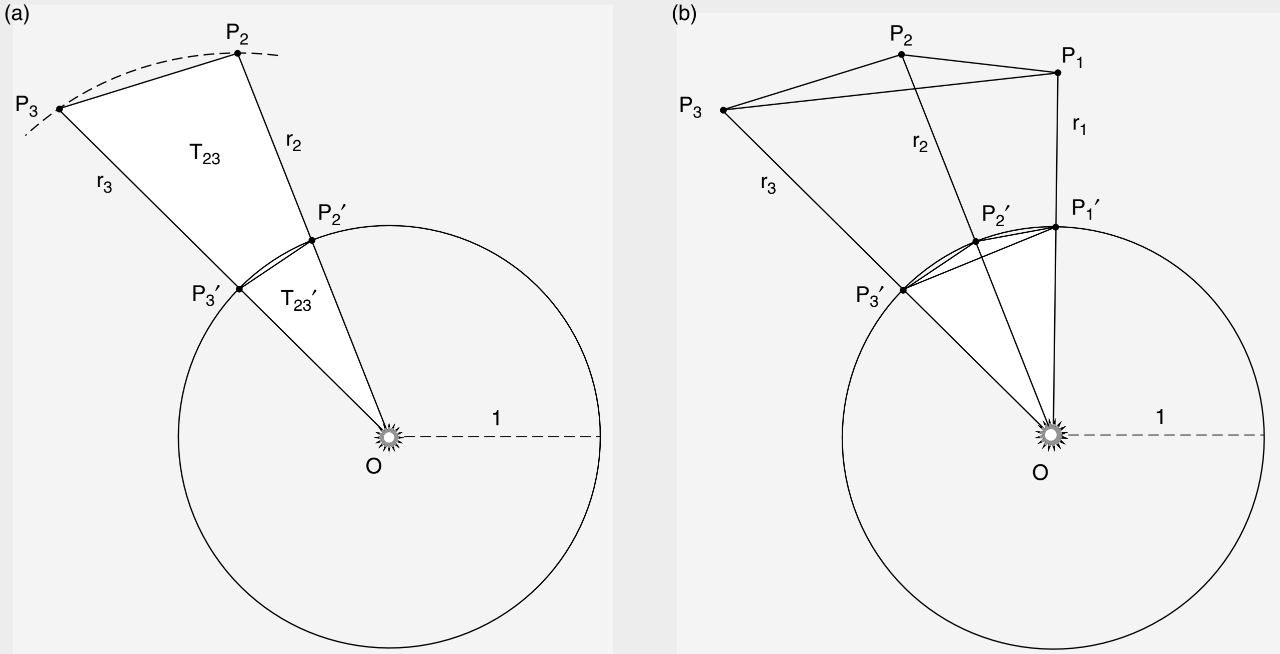
$$r_2 \times T_{13} = r_1 \times r_2 \times r_3 \times T_{13}' .$$

Similarly for T_{12}. In each case, the product of all three radii is a *common factor.* Taking that common factor into account, we can now "translate" **Equation (1)** in terms of the smaller triangles, into

$$T_{123} = \frac{(r_1 \times r_2 \times r_3) \times (T_{23}' - T_{13}' + T_{12}')}{h} . \qquad \textbf{(3)}$$

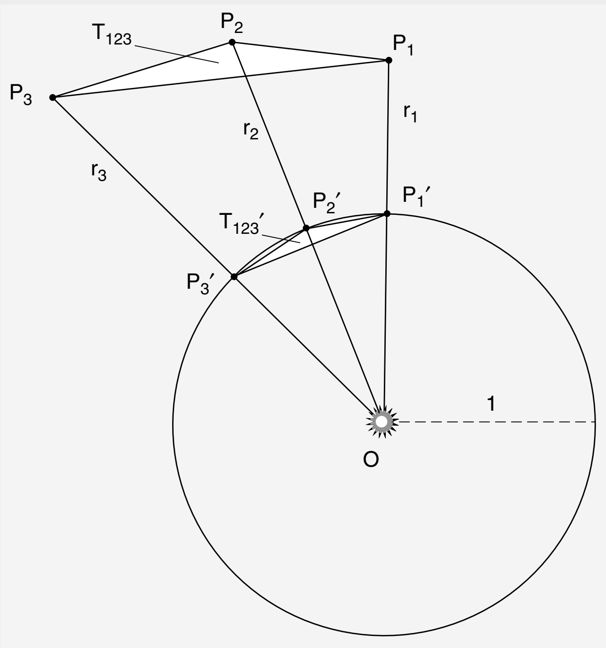
Note that the new triangles, entering into this "distilled" relationship, have the *same apex angles* at the sun, as the original triangles, but the lengths of the radial sides have all been reduced to 1. **(Figure 14.9b)**

FIGURE 14.9. *"Reduction" of relationships on non-circular orbit to relationships in a circle. (a) The area of triangle T_{23}', obtained by projecting P_2 and P_3 onto the circle of unit radius, is equal to $T_{23}/(r_2 \times r_3)$. (b) Similarly for triangles T_{12}' and T_{13}'. The original apex angles at the sun are preserved, but the lengths are all reduced to 1.*



To put it in another way: We have "projected" the Ceres orbit onto the unit circle in Figure 14.9, by central projection relative to O; the triangles T_{23}', T_{13}', T_{12}' are formed in the same way as the old ones, but using instead the points P_1', P_2', P_3' on the unit circle, which are the images of Ceres' positions P_1, P_2, P_3 under that projection. The magnitude expressed as $T_{23}' - T_{13}' + T'_{12}$, is just the triangle between P_1', P_2', P_3' on the unit circle. Using T_{123}' to denote that new "triangular differential" inscribed in the unit circle, our latest result is

FIGURE 14.10. *Triangular area T_{123}', inscribed in the unit circle, depends only on the angles subtended at the sun (O).*



$$T_{123} = \frac{(r_1 \times r_2 \times r_3) \times T_{123}'}{h}. \qquad \textbf{(4)}$$

Keep in mind our earlier conclusion [**Equation (2)**], that h is the product of the sectors S_{12} and S_{23}, divided by π^2 and the product of the elapsed times.

What we have accomplished by this analysis is, in effect, to reduce the geometry of an arbitrary conic-section orbit, to that of a simple circular orbit. Indeed, the vertices of the triangular area T_{123}', the positions P_1', P_2', P_3', all lie on the unit circle, and the area T_{123}' depends only on the *angles* subtended by Ceres' positions at the sun. **(Figure 14.10)**

Now, we can apply the same theorem of Classical Greek geometry, as we earlier evoked for the case of a circular orbit. The area of the triangle is equal to the product of the sides, divided by four times the radius of the circle upon which the vertices lie (in this case, the unit circle). In this case the result is

$$T_{123}' = \frac{(\text{length } P_1'P_2' \times \text{length } P_2'P_3' \times \text{length } P_3'P_1')}{4} . \quad \textbf{(5)}$$

So far, we have employed rigorous geometrical relationships throughout. To the extent the orbital motion of Ceres is governed by the Kepler-Gauss constraints, and to the extent the theorems of Classical Greek geometry are valid for elementary spatial relationships on the scale of our solar system, our calculation of T_{123}' and T_{123} is precisely correct.

At this point, Gauss evokes some apparently rather crude estimates for the factors which go into the product for T_{123}'. In fact, they are the same sort of crude approximations, which we attempted in our original attempt to calculate the Earth-Ceres distance. If that sort of approximation introduced an unacceptable degree of error *then,* how dare we to do the same thing, *now?*

Remember, we had determined that the "differential" T_{123}, whose magnitude we now wish to estimate, accounts for nearly all of the percentual error, which our earlier approach would have introduced into our calculation of the Earth-Ceres distance, by ignoring the discrepancy between the orbital sectors and the triangular areas. Gauss remarked, in fact, that the discrepancies corresponding to pairs of *adjacent* positions, namely between S_{12} and T_{12} and between S_{23} and T_{23}, are practically an order of magnitude smaller than the discrepancy between S_{13} and T_{13}, i.e., the one corresponding to the extreme pair of positions, which span the relatively largest arc on the orbit. (Figures 12.2 and 14.4) On the other hand, the difference between S_{13} and T_{13}, consists of T_{123} together with the small differences $S_{12}–T_{12}$ and $S_{23}–T_{23}$. As a result, T_{123} supplies the *approximate size of the "error"* in our earlier approach, up to quantities an order of magnitude smaller.

An "error" introduced in an approximate value for T_{123}, thus has the significance of a "differential of a differential." In numerical terms, it will be at least one order of magnitude smaller—and the final result of our calculation of Ceres at least an order of magnitude more precise—than the error in our original approach, which ignored the "differential" altogether.

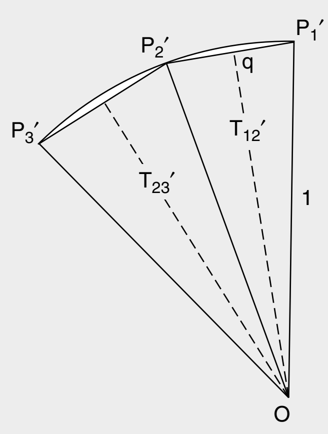
Also remember the following: As a geometrical magnitude, T_{123} measures the effect of curvature of the planetary orbit over the interval from P_1 to P_3. The *relative* crudeness of the approximations we shall introduce now, concern the order of magnitude of the *change in local curvature* over that interval. But once these "second-order" approximations have served their purpose, permitting us to obtain a *tolerable first approximation* for the Earth-Ceres distance, we shall immediately turn around, and use the *coherence* of a first-approximation Keplerian orbit, to eliminate nearly the entire error introduced thereby.

Finishing Up the Job

Turn now to the final estimation of the "differential" T_{123}. Our immediate goal is to eliminate all but the *most essential factors* entering into the function for T_{123}, developed above, and relate everything as far as possible to the known, elapsed times.

First of all, remember that P_1', P_2', P_3' lie on the unit circle; the segments $P_2'P_1'$, $P_3'P_2'$, $P_3'P_1'$ are thus chords of arcs on the unit circle, at the same time form the *bases* of the rather thin isosceles triangles, with common apex at O, whose areas we have designated T_{12}', T_{23}', and T_{13}'. **(Figure 14.11)** The altitudes of those triangles are the radial lines connecting O with the midpoints of the respective chords. Now, if the apex angles at O are relatively small, the gap between the chords and the circular arcs will be very small, and the radial lines to the midpoints of the chords will be only very slightly shorter than the radius of the circle (unity). Let us, by way of approximation, take the altitudes of the triangles to be equal to unity. In that case, the areas of the triangles will be half the lengths of their bases, or, conversely,

FIGURE 14.11. *Estimating the areas of triangles T_{12}', T_{23}', T_{13}'. The area of a triangle is equal to (half the base) × (the altitude). Taking $P_1'P_2'$ as the base of triangle T_{12}', the corresponding altitude is the length of the dashed line Oq. When P_1' and P_2' are close together, Oq will be only very slightly smaller than the radius of the circle, which is 1. Hence, the area of T_{12}' will be very nearly (1/2) × ($P_1'P_2'$). Similarly, area $T_{23}' \simeq$ (1/2) × ($P_2'P_3'$), and area $T_{13}' \simeq$ (1/2) × ($P_1'P_3'$).*



$P_2'P_1'$ = (very nearly) $2 \times T_{12}'$,

$P_3'P_2'$ = (very nearly) $2 \times T_{23}'$,

$P_3'P_1'$ = (very nearly) $2 \times T_{13}'$.

Applying these approximations to **Equation (5)**, we find that T_{123}' is approximately equal to

$$\frac{(2 \times T_{12}') \times (2 \times T_{23}') \times (2 \times T_{13}')}{4}, \qquad \textbf{(6)}$$

or twice the product of T_{12}', T_{23}', and T_{13}'.

This is a very elegant result. But, what does it tell us about the relationship of T_{123} to T_{12}, T_{23}, and T_{13} on the original, non-circular orbit? Remember how we obtained the triangular areas entering into the above product. In numerical values, T_{12}', T_{23}', and T_{13}' are equal to the quotients of $T_{12}/(r_1 \times r_2)$, $T_{23}/(r_2 \times r_3)$, $T_{13}/(r_1 \times r_3)$, respectively. Expressed in terms of those original triangles, our approximate value for T_{123}' becomes

$$2 \times \frac{T_{12} \times T_{23} \times T_{13}}{(r_1 \times r_2) \times (r_2 \times r_3) \times (r_1 \times r_3)}. \qquad \textbf{(7)}$$

Note, that each of r_1, r_2, r_3 enters into the long product exactly twice.

Finally, use this approximate value for T_{123}', to compute T_{123}, according to relationship **(4)** above, noting that half of the radii factors cancel out in the process:

$$T_{123} = \frac{(r_1 \times r_2 \times r_3) \times T_{123}'}{h} \qquad \text{[by } \textbf{Equation (4)}\text{]}$$

= [very nearly, by **Equation (7)**]

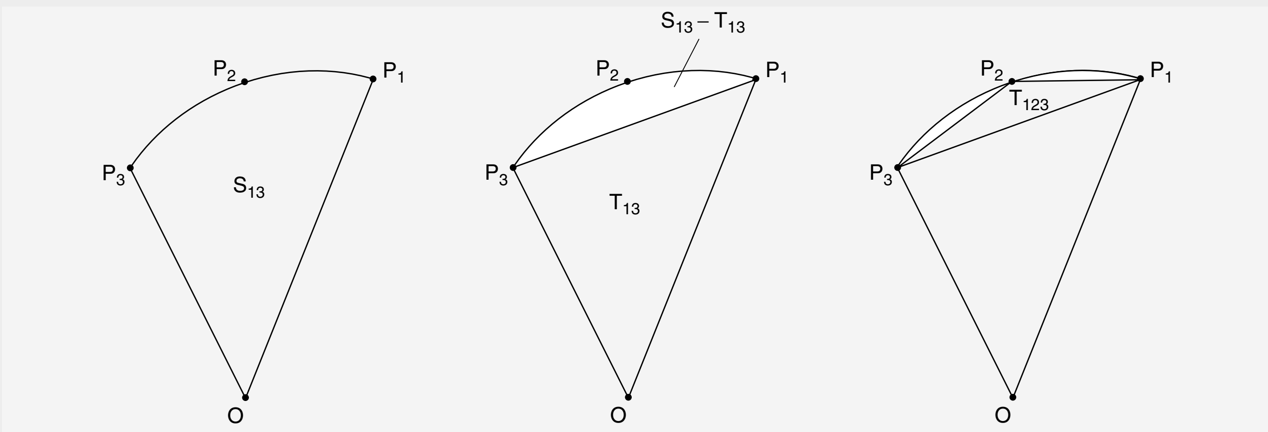
$$2 \times \frac{(T_{12} \times T_{23} \times T_{13}) / (r_1 \times r_2 \times r_3)}{h}. \qquad \textbf{(8)}$$

A bit of bookkeeping is required, as we take into account our calculation of h, as the quotient of the product of S_{12} and S_{23}, divided by π^2 times the product of the corresponding elapsed times. [**Equation (2)**] The result of *dividing* by h, is to *multiply* by π^2 and the elapsed times, and *divide* by the product of the sectors. Assembling all these various factors together, with **Equation (8)**, our approximate value for T_{123} becomes

$$2 \times \frac{\pi^2 \times (t_2 - t_1) \times (t_3 - t_2) \times T_{12} \times T_{23} \times T_{13}}{S_{12} \times S_{23} \times r_1 \times r_2 \times r_3}. \qquad \textbf{(9)}$$

For reasons already discussed above, we can permit ourselves simplifying approximations at this point, as follows. For a relatively short interval of motion, the sun-Ceres distance does not change "too much." Thus, we can approximate the product $r_1 \times r_2 \times r_3$ by the cube of the second distance r_2, i.e., by the product $r_2 \times r_2 \times r_2$, without introducing a large error *in percentual terms.* Next, observe that T_{12} and T_{23} appear in the numerator, and S_{12} and S_{23} in the denominator, of the quotient we are now estimating. If we simply *equate* the corresponding triangular and sectoral areas—whose discrepancies are practically an order of magnitude less than that between S_{13} and T_{13}—we introduce an additional, but tolerable percentual error into the value of T_{123}. Applying these considerations to **Equation (9)**, we obtain, as our final approximation, the value

FIGURE 14.12. *S_{13} is (to a first order of approximation) very nearly equal to T_{13} + T_{123}.*



$$T_{123} \simeq 2 \times \frac{\pi^2 \times (t_2 - t_1) \times (t_3 - t_2)}{r_2^{\,3}} \times T_{13} \,. \qquad \textbf{(10)}$$

Recall the original motive for this investigation, which was to "get a grip" on the relationship between the sectoral area S_{13} and the triangle T_{13}. What we can say now, by way of a crucially useful approximation, is the following. Since T_{123} makes up nearly the whole difference between the triangle T_{13} and the orbital sector S_{13} **(Figure 14.12)**,

$$S_{13} = \text{(to a first order of approximation) } T_{13} + T_{123} \,,$$

or, stating this in terms of a ratio,

$$\frac{S_{13}}{T_{13}} = \text{(very nearly) } 1 + \frac{T_{123}}{T_{13}} \,.$$

On the other hand, we just arrived in **Equation (10)** at an approximation for T_{123}, in which T_{13} is a factor. Applying that estimate, we conclude that

$$\frac{S_{13}}{T_{13}} \simeq 1 + \left(2 \times \frac{\pi^2 \times (t_2 - t_1) \times (t_3 - t_2)}{r_2^{\,3}} \right) .$$

The hard work is over. We have arrived at the crucial "correction factor," which Gauss supplied to complete his first-approximation determination of Ceres' position. For some one hundred fifty years, following the publication of Gauss's *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections,* astronomers around the world have used it to calculate the orbits of planets and comets. All that remains to be done, we shall accomplish in the next chapter.

—*JT*